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NUMERICAL SOLUTION OF FUNCTIONAL EQUATIONS BY MEANS OF LAPLACE TRANSFORM–IV: NONLINEAR EQUATIONS

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PREFACE

Mathematical studies of biological processes have many nonlinear functional equations. This study shows that the method of successive approximations, coupled with the numerical inversion of Laplace transforms, provides an effective computational approach.

For illustrative purposes a nonlinear differential equation, a nonlinear differential-difference equation and a nonlinear diffusion equation are considered.

See also AD-428 074.

SUMMARY

It is shown that Laplace transform can be effective in the numerical solution of nonlinear functional equations. to p iii

CONTENTS

I. INTRODUCTION

In the previous Memoranda in this series,[1-3] we have demonstrated the effectiveness of the Laplace transform in obtaining the numerical solution of linear functional equations. In I, we considered the renewal equation, in II, differential-difference equations, and in III, the heat equation.

Here we wish to consider the possibility of using numerical inversion of the Laplace transform plus successive approximations to treat nonlinear functional equations. To illustrate our techniques, we consider the nonlinear differential equation

$$u' = -u - u^2, \qquad u(0) = c, \tag{1}$$

the nonlinear differential-difference equation

$$u'(t) = -u(t-1) - u^2(t), \qquad u(t) = c, \; -1 \le t \le 0, \tag{2}$$

and the nonlinear heat equation

$$\begin{gathered} k(x)u_t = u_{xx} + bu^2 \\ u(x,0) = c \sin \pi x, \qquad u(0,t) = u(1,t) = 0 \, . \end{gathered} \tag{3}$$

Numerical results are given, as well as times of execution.

II. DIFFERENTIAL EQUATIONS

To illustrate the general approach in its simplest form, let us take the nonlinear differential Eq. (1). Taking Laplace transforms, we have

$$L(u') = - L(u) - L(u^2), \tag{4}$$

$$\begin{aligned} (s+1)\, L(u) &= c - L(u^2), \\ L(u) &= \frac{c - L(u^2)}{(s+1)} . \end{aligned} \tag{5}$$

We solve this equation numerically by means of successive approximations plus numerical quadratures in the following fashion. The function $u_o(t)$ is taken as the solution of the equation

$$L(u) = \frac{c}{s + 1} . \tag{6}$$

Ignoring the fact that we can find $u(t)$ explicitly, we use the quadrature techniques presented in I, II, and III to obtain the values $\{u_o(r_i)\}$.

Next we write

$$L(u_1) = \frac{c - L(u_o^2)}{s + 1} . \tag{7}$$

To evaluate $L(u_o^2)$, we employ the same quadrature formula

$$L(u_o^2) = \sum_{i=1}^{N} w_i\, u_o^2(r_i) r_i^{s-1} . \tag{8}$$

We now determine u_1 using quadrature techniques and the explicit inversion formula.

Continuing in this fashion, we generate a sequence of functions $\{u_n(t)\}$ evaluated at the points $t_i = -\log r_i$, $i=1,2,\ldots,N$, with

$$L(u_n) = \frac{c - L(u_{n-1}^2)}{(s+1)} \tag{9}$$

Naturally, the procedure will converge only if $|c|$ is sufficiently small. Subsequently, we shall discuss what techniques can be used in the general case.

III. NUMERICAL RESULTS

Ten successive approximations for each value of c can be accomplished in 6 sec of execution time on the IBM 7090. Graphs of the solutions are given in Figs. 1 and 2.

IV. DIFFERENTIAL-DIFFERENCE EQUATIONS

Similarly, starting with Eq. (2), we have

$$L(u'(t)) = -L(u(t-1)) - L(u^2(t)), \tag{10}$$

whence

$$L(u)\ (s+e^{s}) = c - \int_0^1 ce^{-st}\,dt - L(u^2). \tag{11}$$

Hence, we compute u_o by means of

$$L(u_o) = \frac{c - \int_0^1 ce^{-st}\,dt}{s + e^{-s}} \tag{12}$$

and then u_1 from

$$L(u_1) = \frac{c - \int_0^1 ce^{-st}\,dt - L(u_o^2)}{(s+e^{-s})} \tag{13}$$

We see that despite the great increase in complexity of a differential-difference equation over an ordinary differential equation, the computational time and effort using this procedure is virtually the same for both types of equations. Once again, we have convergence of this iteration procedure only if $|c|$ is sufficiently small.

V. NUMERICAL RESULTS

Fifteen seconds of execution time are required for ten successive approximations. Graphs of the numerical results are given in Fig. 3. For some reasons which we have not as yet understood, all of the values obtained following the procedure given above are in excellent agreement with the actual solution, except for the very last value of t. This will be investigated at a later time.

VI. HEAT EQUATION

Consider next the nonlinear heat equation

$$\begin{aligned} k(x)u_t &= u_{xx} + b\, g(u), \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= h(x). \end{aligned} \tag{14}$$

Using the Laplace transform, we readily obtain

$$L(u)'' - s\, k(x)\, L(u) = -\, k(x)h(x) - b\, L(g(u)), \tag{15}$$

with the two-point boundary condition

$$L(u) = 0, \qquad x = 0,1. \tag{16}$$

Here $'$ denotes d/dx.

As above, we employ successive approximations. Let u_o be the solution of

$$\begin{aligned} L(u_o)'' - s\, k(x)L(u_o) &= -\, k(x)h(x), \\ L(u_o) = 0, \qquad x &- 0,1. \end{aligned} \tag{17}$$

First we solve the differential equation numerically for a set of s-values, $s = 1,2,\ldots,N$, and then we use the Laplace inversion technique. Storing the values $\{u_o(x,r_i)\}$ for $x = 0,\Delta,2\Delta,\ldots,M\Delta=1$, we then solve the equation

$$\begin{aligned} L(u_1)'' - s\, k(x)L(u_1) &= -\, k(x)h(x) - b\, L(g(u_o)). \\ L(u_1) = 0, \qquad x &= 0,1. \end{aligned} \tag{18}$$

Continuing in this way, we obtain a sequence of values $\{u_n(x,r_i)\}$.

VII. NUMERICAL RESULTS

Five successive approximations require 6 min 10 sec of IBM 7090 execution time, and the numerical agreement was quite satisfactory. Results of this method are shown in Figs. 4 and 5.

VIII. CONVERGENCE ASPECTS

From the standpoint of rapidity of convergence, we would prefer to use quasilinearization procedures (cf. Refs. 4-6). For example, in treating the differential equation of Section II, we would prefer to use successive approximations in the form

$$L(u_n') = - L(u_n) - L(u_{n-1}^2 + 2u_{n-1}(u_n-u_{n-1})). \tag{19}$$

This, however, complicates the matrix inversion aspects. Consequently, we feel that it is better to use the simpler, but less efficient approximation technique.

There is also the problem that the initial function, the solution of the linear equation, may have too large a norm for convergence. Provided that the solution of the nonlinear problem exists, we can often use extrapolation techniques (cf. Ref. 7).

For example, we can replace

$$k(x)u_t = u_{xx} + u^2 \tag{20}$$

by

$$k(x)u_t = u_{xx} + bu^2, \tag{21}$$

and use the new procedures for small b. The solution $u(x,t) = u(x,t,b)$ can then be determined for larger values of b by using extrapolation formulas.

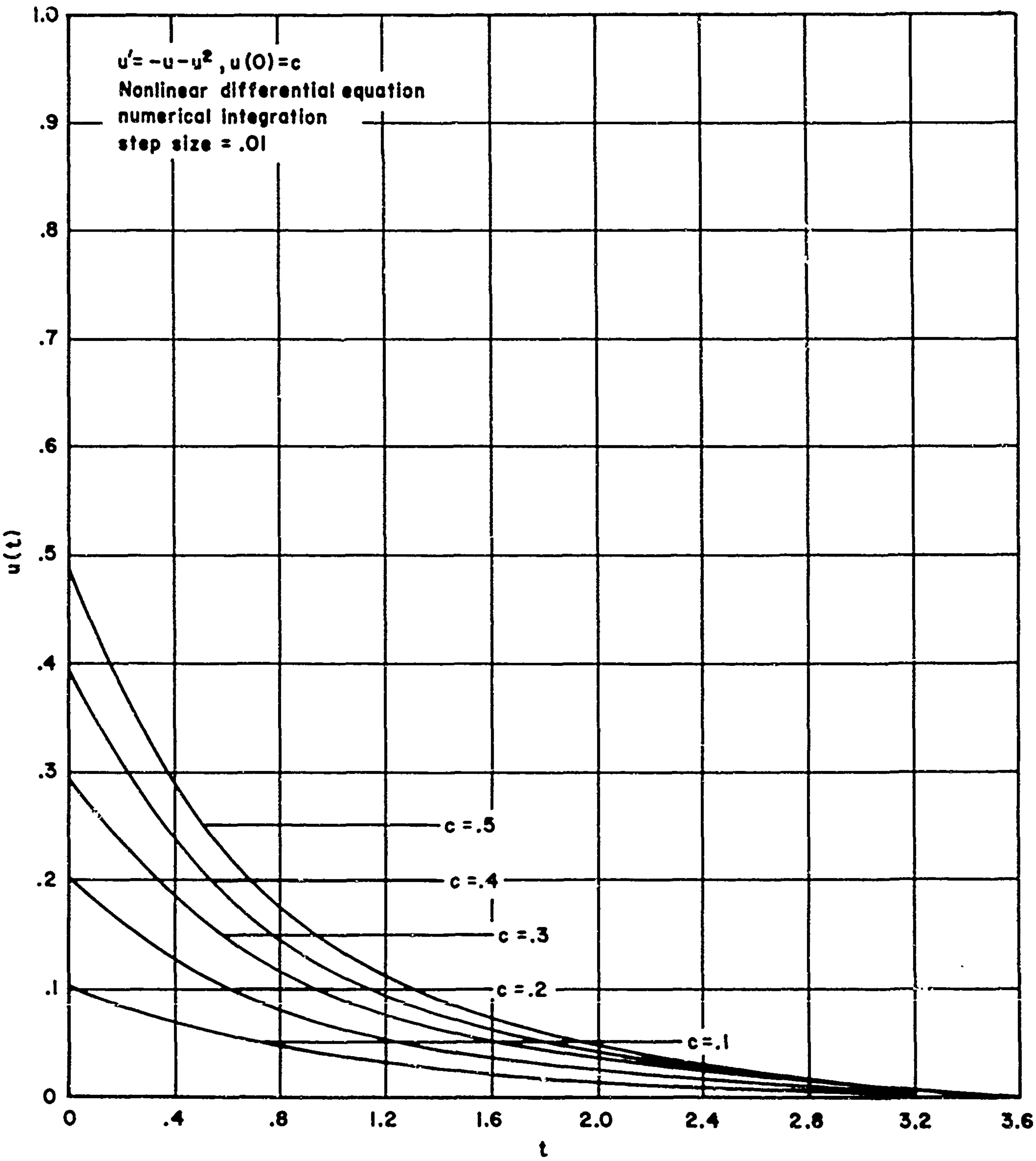

u′= −u−u², u(0)=c
Nonlinear differential equation
numerical integration
step size = .01
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
u(t)
c =.5
c =.4
c =.3
c =.2
c =.1
0
.4
.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
t

Fig. 1

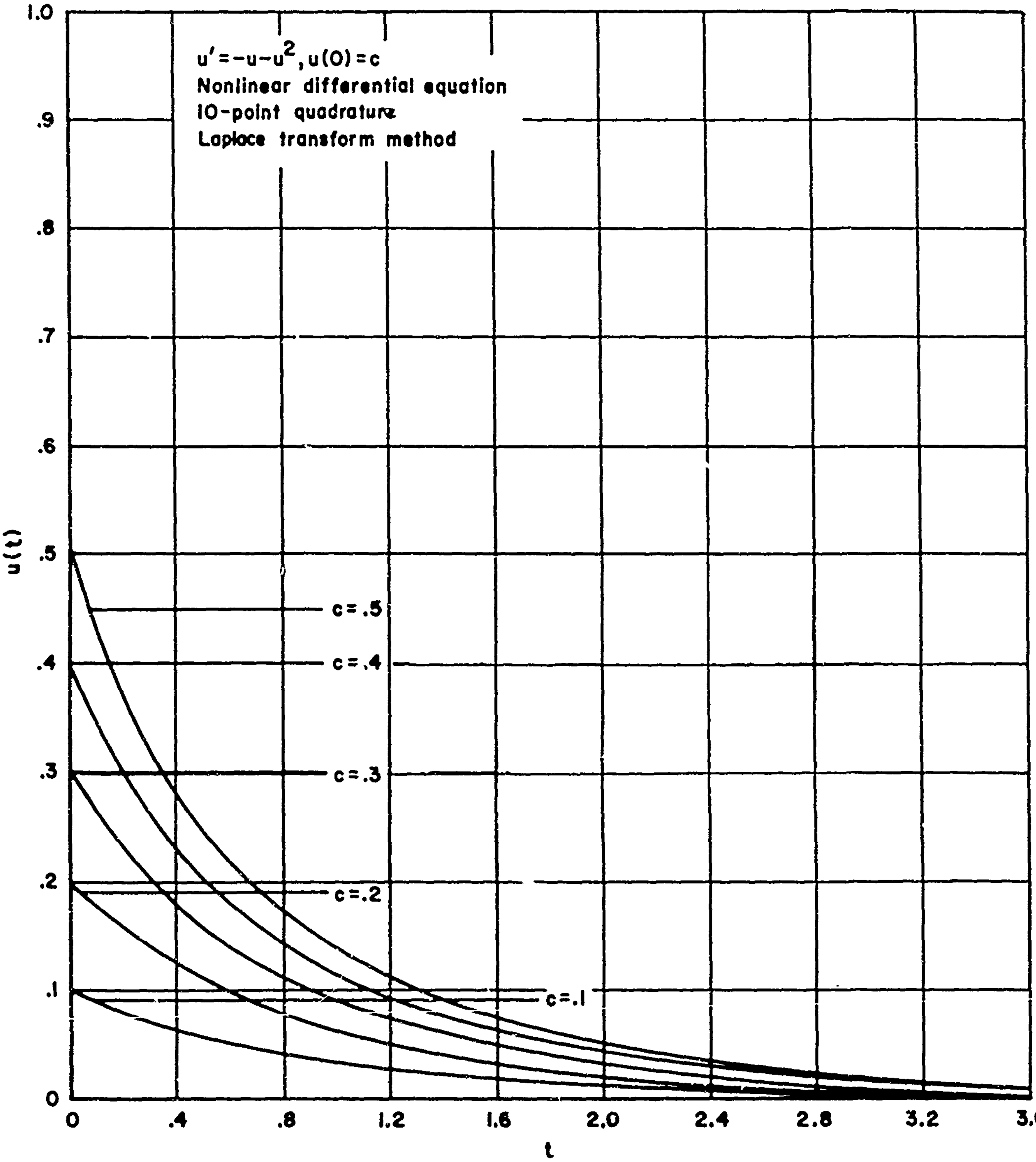
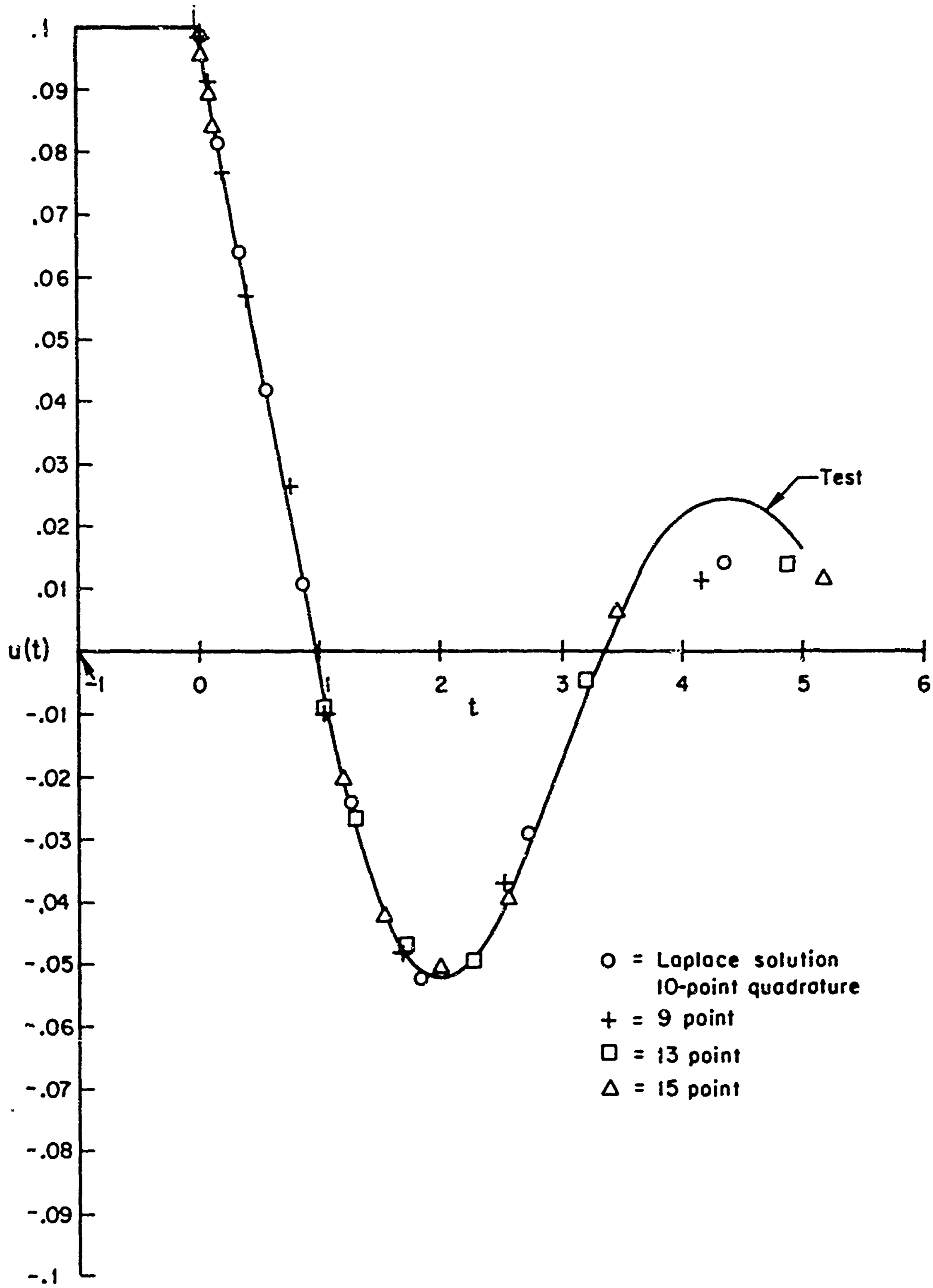

u′=−u−u2, u(0)=c
Nonlinear differential equation
10-point quadrature
Laplace transform method
u(t)
t
c=.5
c=.4
c=.3
c=.2
c=.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
.4
.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6

Fig. 2



u(t)
.1
.09
.08
.07
.06
.05
.04
.03
.02
.01
-.01
-.02
-.03
-.04
-.05
-.06
-.07
-.08
-.09
-.1
-1
0
1
2
3
4
5
6
t
Test
O = Laplace solution 10-point quadrature
+ = 9 point
□ = 13 point
Δ = 15 point

Fig. 3

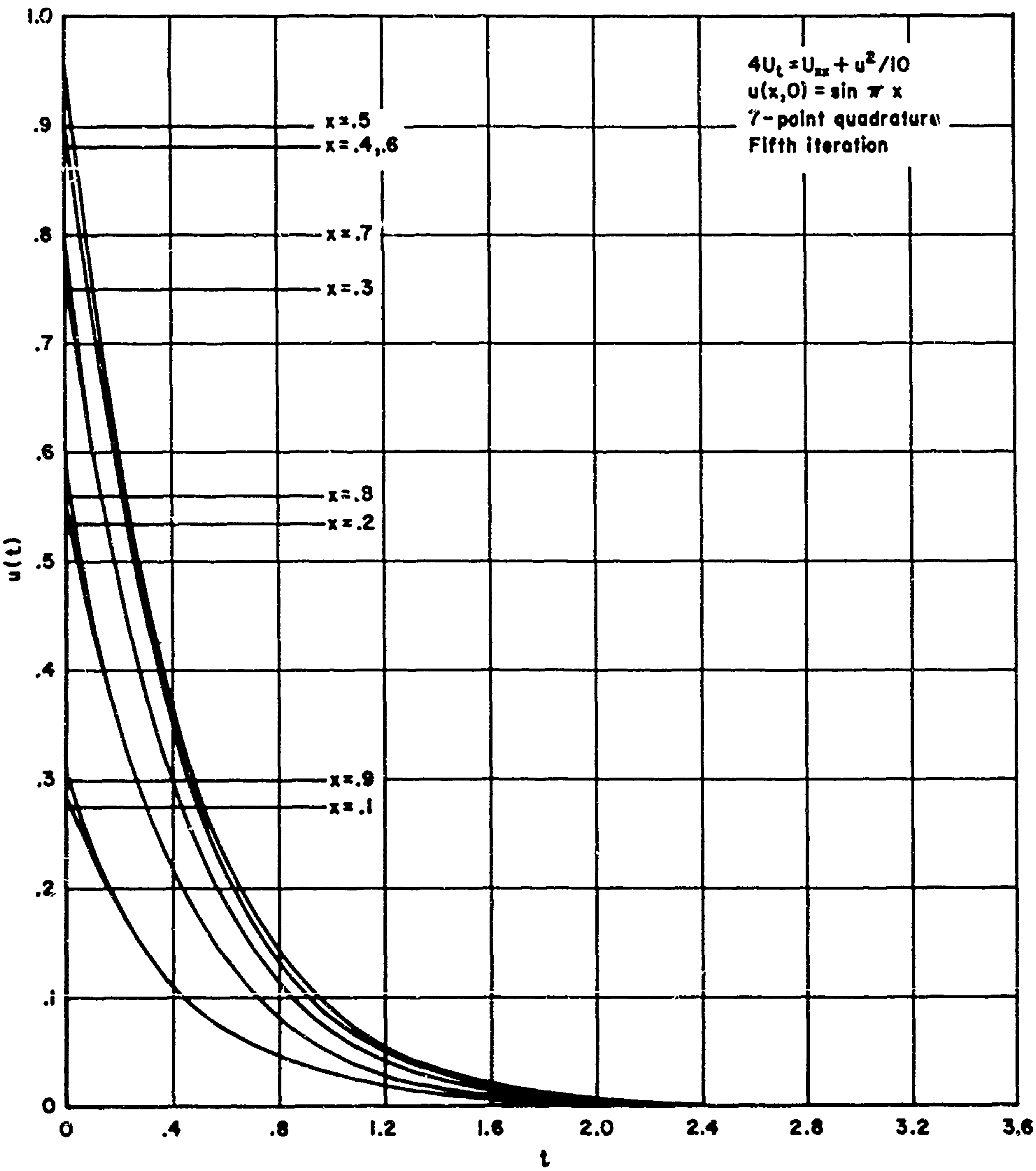

$4U_t = U_{xx} + u^2/10$
$u(x,0) = \sin \pi x$
7-point quadrature
Fifth iteration
x = .5
x = .4, .6
x = .7
x = .3
x = .8
x = .2
x = .9
x = .1
u(t)
t
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
.4
.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6

Fig. 4

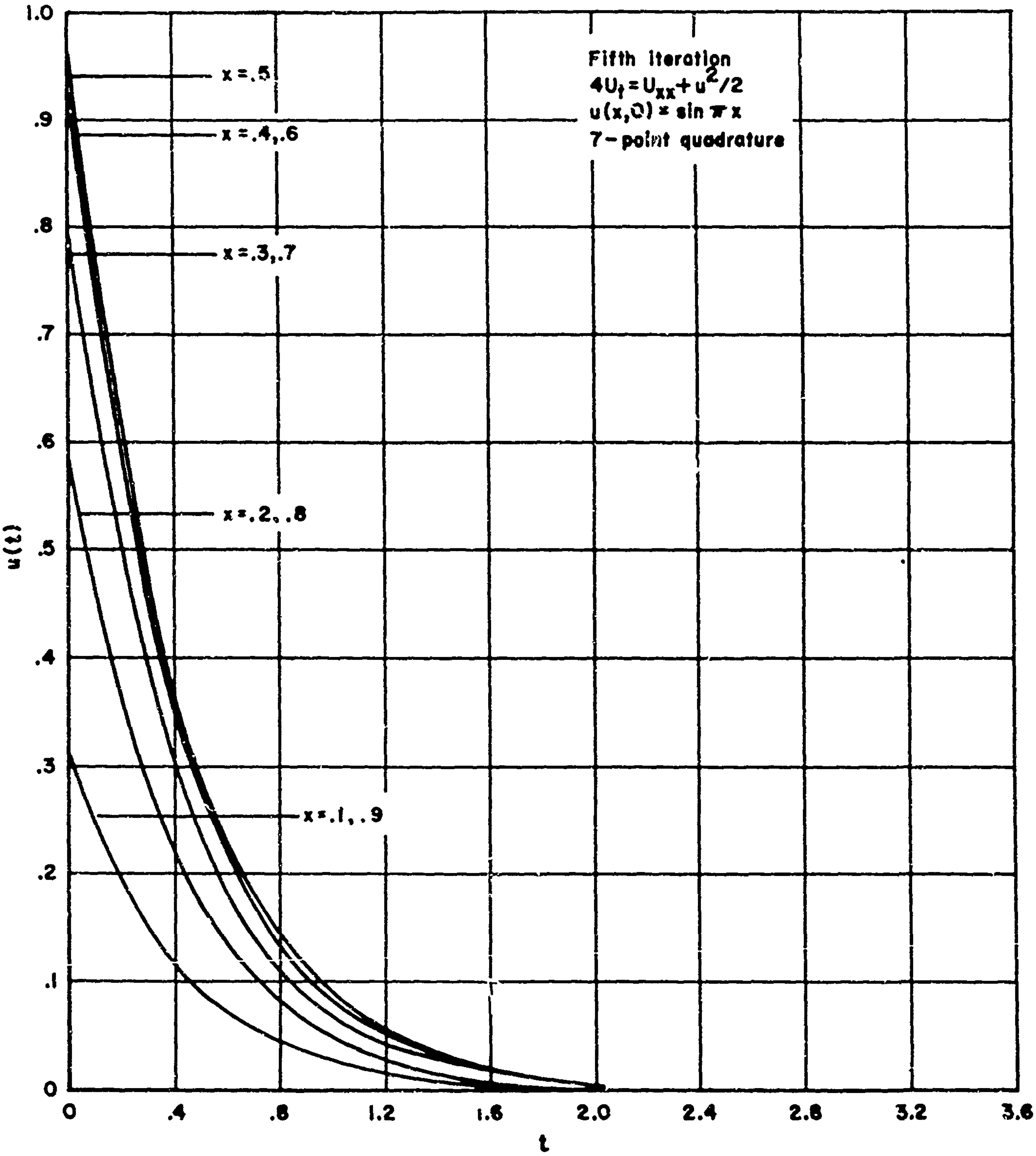

Fifth iteration
$4U_t = U_{xx} + u^2/2$
$u(x,0) = \sin \pi x$
7-point quadrature
x = .5
x = .4, .6
x = .3, .7
x = .2, .8
x = .1, .9
u(t)
t
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
.4
.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6

Fig. 5

```
*      XEQ
*      LIST
*      LABEL
CNONLIN
C     NON LINEAR DIFFERENTIAL EQUATION
D     DIMENSION R(15),W(15),A(15,15),T(15),U(15),B(15)
   10 FORMAT(I12)
   11 FORMAT(1H1,4X11HDIMENSION =,I3)
   13 FORMAT(//5X32HROOTS                                  /(5X6E20.8))
   15 FORMAT(//5X32HWEIGHTS                                /(5X6E20.8))
   16 FORMAT(//5X16HEXPLICIT INVERSE//)
   17 FORMAT(5X6E20.8)
  101 FORMAT(//5X32HT = -LOG R                             /(5X6E20.8))
  102 FORMAT(E12.8)
  103 FORMAT(//10X4HC = ,F5.2)
   32 FORMAT(//5X32HLAPLACE TRANSFORM                      /(5X6E20.8))
   21 FORMAT(//5X32HINITIAL APPROXIMATION OF U(I)          /(5X6E20.8))
   42 FORMAT (//5X23HVALUE OF U(I) AT STAGE , I2           /(5X6E20.8))
C
C                 I   INPUT
C
      READ 10, NA
      READ 10,N
      PRINT 11,N
      DO 12 I=1,N
D     CALL DBLRED(X)
D  12 R(I)=X
      PRINT 13, (R(I),I=1,N)
      DO 14 I=1,N
D     CALL DBLRED(X)
D  14 W(I)=X
      PRINT 15, (W(I),I=1,N)
D     CALL EXPINV (N,R,W,A)
      PRINT 16
      DO 18 I=1,N
   18 PRINT 17, (A(I,J),J=1,N)
      DO 19 I=1,N
D  19 T(I)=-LOGF(R(I))
      PRINT 101, (T(I),I=1,N)
  100 READ 102, C
      PRINT 103, C
C
C                II   INITIAL APPROXIMATION
C
      DO 20 I=1,N
D  20 U(I)=C*R(I)
      PRINT 21, (U(I),I=1,N)
C
C               III   TRANSFORM
C
      DO 30 K=1,NA
      DO 31 I=1,N
D     S=I
```

REFERENCES

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```
      B(I)=C/(S+1.)
      DO 31 J=1,N
   31 B(I)=B(I)-(W(J)*R(J)**(I-1)*U(J)**2)/(S+1.)
      PRINT 32, (B(I),I=1,N)

                  IV  NEW APPROXIMATION

      DO 41 I=1,N
      U(I)=0.0
      DO 41 J=1,N
   41 U(I)=U(I)+A(I,J)*B(J)
      PRINT 42, K,(U(I),I=1,N)
   30 CONTINUE
      GO TO 100
      END
```